Math 259A Lecture 2 Notes

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1 Introduction to C^*-Algebras

1.1 Recap

Recall: We are interested in the following objects.

Definition 1.1. A ***-algebra** M is an algebra with an involution $*$ (called the **adjoint**) such that if $T \in M$, then $T^* \in M$.

Definition 1.2. A **von Neumann algebra** $M \subseteq B(H)$ is a *-algebra of operators on a Hilbert space with $1 = \mathrm{id}_H \in M$ which is closed in the weak operator topology.

Definition 1.3. A **C^*-algebra** is a *-algebra of operators $M_0 \subseteq B(H)$ with $1_{M_0} = \mathrm{id}_H$ which is closed in the operator norm.

Remark 1.1. Since the weak operator topology is weaker than the norm topology, von Neumann algebras are C^*-algebras.

Definition 1.4. A Banach algebra is a Banach space with multiplication such that $\|xy\| \leq \|x\|\|y\|$.

We will aim to prove the following.

Theorem 1.1. *If M is a Banach algebra (with 1_M) and with an involution * satisfying $\|x^*x\| = \|x\|^2$ for all $x \in M$, then there is a injective, isometric *-algebra morphism $\theta : M \to B(H)$. In other words, any algebra satisfying these axioms is a concrete C^*-algebra.*[1]

1.2 Involutive algebras

Definition 1.5. If M is an algebra (over $\mathbb{C}$), then an **involution** on M is a map $* : M \to M$ satisfying

[1]We can consider these to be the "abstract C^*-axioms."

1. $(\lambda x)^* = \overline{\lambda} x^*$
2. $(x+y)^* = x^* + y^*$
3. $(xy)^* = y^* x^*$
4. $(x^*)^* = x$.

Example 1.1. $B(H)$ has the adjoint map as an involution.

Example 1.2. If X is compact, $C(X)$ is an algebra with the involution of complex conjugation given by $\overline{f}(x) = \overline{f(x)}$. If we take $C_0(X)$ where X is only locally compact, then we still get an algebra, but it does not have an identity.

Example 1.3. Let G be a group. Then $L^1(G)$ is an algebra with the product $f \cdot g$ of convolution. We have the involution $f^*(g) = \overline{f(g^{-1})}$.

Proposition 1.1. *The adjoint satisfies the following properties:*

1. $1^* = 1$.

2. If x is invertible ($x \in \mathrm{Inv}(M)$), then $x^* \in \mathrm{Inv}(M)$, and $(x^*)^{-1} = (x^{-1})^*$.

Definition 1.6. If $x = x^*$, we call x **Hermitian**. We denote the set of Hermitian elements as $M_h = \{x \in M : x = x^*\}$.

Definition 1.7. An element $x \in M$ is **normal** if $x^* x = x x^*$.

In this case, the $*$-algebra generated by x is commutative.

Definition 1.8. An element $x \in M$ is **unitary** if $x^* x = x x^* = 1$ (i.e. x is invertible an $x^{-1} = x^*$. We denote $U(M)$ to be the set of unitary elements, which is a subgroup of $\mathrm{Inv}(M)$.

Definition 1.9. An element $x \in M$ is an **isometry** if $x^* x = 1$.

Remark 1.2. In general, this does not necessarily mean that x is unitary. For example, we can take the map $x : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $(x_0, x_1, x_2, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$.

Definition 1.10. An element $x \in M$ is an **orthogonal projection** if $x^2 = x = x^*$.

Definition 1.11. An element $x \in M$ is a **partial isometry** if $x^* x$ and $x x^*$ are projections.

Proposition 1.2. *We can always decompose $x = \mathrm{Re}\, x + i\,\mathrm{Im}(x)$, where $\mathrm{Re}\, x, \mathrm{Im}\, x$ are Hermitian via*

$$\mathrm{Re}(x) = \frac{x + x^*}{2}, \qquad \mathrm{Im}(x) = \frac{x - x^*}{2i}.$$

Definition 1.12. If $x \in M$, the **spectrum** of x is the set $\mathrm{Spec}(x) = \{x \in \mathbb{C} : \lambda 1 - x \text{ is not invertible in } M\}$. We also call $\rho(X) = \{\lambda : \lambda 1 - x \text{ is invertible}\}$ the **resolvent** of x.

Proposition 1.3. $\mathrm{Spec}(x^*) = (\mathrm{Spec}(x))^*$, *and* $\mathrm{Spec}(x^{-1}) = (\mathrm{Spec}(x))^{-1}$.

Definition 1.13. Functionals on an involutive[2] algebra M are linear maps $\varphi : M \to \mathbb{C}$. The involution on functionals is given by $\varphi^*(x) = \overline{\varphi(x^*)}$.

1.3 Normed involutive algebras

Definition 1.14. A **normed involutive algebra** M is an involutive algebra with a norm such that $\|xy\| \leq \|x\|\|y\|$ and $\|x^*\| = \|x\|$ for all $x \in M$. This is a **Banach algebra** if M is complete.

Definition 1.15. If M is a Banach space, we denote the **dual space** M^* to be the set of continuous linear functionals on M.

Proposition 1.4. $\|\varphi^*\| = \|\varphi\|$ *for all* $\varphi \in M^*$. *Also, if* $\varphi = \varphi^*$, *then* $\|\varphi|_{M_h}\| = \|\varphi\|$.

Notation: If X is a Banach space and $r > 0$, then we denote the closed unit ball as $(X)_r := \{x \in X : \|x\| \leq r\}$.

Definition 1.16. A Banach algebra M with involution satisfying $\|x^*x\| = \|x\|^2$ for all $x \in M$ is called an (abstract) C^*-algebra. This condition is called the **C^*-axiom**.

Remark 1.3. It is enough to show that $\|x^*x\| \geq \|x\|^2$ for all x.

Proposition 1.5. $\|x\| = \sup_{y \in (M)_1} \|xy\|$. *This gives us an isometric embedding of* $M \to B(M)$ *given by* $x \mapsto L_x$, *where* $L_x(y) = xy$.

Proposition 1.6. *If* $M \neq 0$, *then* $\|1\| = 1$.

Proposition 1.7. *For any* $u \in U(M)$, $\|u\| = 1$.

1.4 Spectra in Banach algebras

Definition 1.17. The **spectral radius** of x is $R(x) = \sup\{|\lambda| : \lambda \,in\, \mathrm{Spec}(x)\}$.

Proposition 1.8. $R(x) \leq \|x\|$.

If M is a Banach algebra, $x \in M$ and f is an entire function on $\mathbb{C}$, then $f(x) = \sum_{n=0}^{\infty} a - nx^n$ makes sense.

[2] We call them involutive because using the term *-algebra makes people strictly think of operator algebras.

Example 1.4. We can define $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$.

Proposition 1.9. *If M has an involution and $h \in M_h$, then* $\exp(ih) = \exp(-ih)$.

Proposition 1.10. *Let M be an involutive Banach algebra, Then*

1. *If $h = h^*$, then* $\exp(ih) \in U(M)$.
2. *If $u \in U(M)$, then* $\operatorname{Spec}(u) \subseteq \mathbb{T}$.
3. *If $h = h^*$, then* $\operatorname{Spec}(h) \subseteq \mathbb{R}$.

Proof.

1. $u = \exp(ih)$ has $u^* = \exp(-ih)$ as its inverse.
2. $\operatorname{Spec}(u) = (\operatorname{Spec}(u^{-1})^{-1}$, and $\|R(u)\| \leq \|u\|$ and $\|R(u^{-1})\| \leq \|u^{-1}\|$.
3. $\operatorname{Spec}(h) = \operatorname{Spec}(h^*) = \overline{\operatorname{Spec}(h)}$. □

Lemma 1.1. *Let M be a Banach algebra, and let $x \in M$ iwth $\|1 - x\| < 1$. Then x is invertible, and $\|x^{-1}\| \leq 1/(1 - \|1 - x\|)$.*

Proof. The series $y = \sum_{n=0}^{\infty}(1 - x)^n$ is convergent in norm and hence makes sense in M. Then

$$xy = (1 - (1 - x)) \sum_{n=0}^{\infty}(1 - x)^n = \lim(1 - (1 - x)^{n+1}) = 1,$$

so y is an inverse for x. □

Corollary 1.1. $\operatorname{Inv}(M)$ *is open, and the map* $\operatorname{Inv}(M) \to \operatorname{Inv}(M)$ *sending $x \mapsto x^{-1}$ is continuous.*

Proof. Let x be invertible, and let $\|y - x\| \leq 1/\|x^{-1}\|$. Then

$$\|x^{-1}y - 1\| \leq \|x^{-1}\|\|y - x\| < 1,$$

so $x^{-1}y$ is invertible by the lemma. So y is invertible.

Continuity follows from $x^{-1} - y^{-1} = x^{-1}(y - x)y^{-1}$. □

Corollary 1.2. $\operatorname{Spec}(x) \subseteq (M)_{\|x\|}$.

Proof. If $|\lambda| > \|x\|$, then $1 > \|\lambda^{-1}x\|$, so $1 - \lambda^{-1}x$ is invertible by the lemma. So $\lambda - x$ is invertible. So $\lambda \notin \operatorname{Spec}(x)$. □

Theorem 1.2. $\operatorname{Spec}(x)$ *is compact and nonempty.*

Proof. $\mathrm{Spec}(x)$ is closed by continuity of $y \mapsto y^{-1}$. It is bounded, so it is compact. To show that $\mathrm{Spec}(x) \neq \varnothing$, let $F : \rho(x) \to M$ be $F(\lambda) = (\lambda 1 - x)^{-1}$. We claim that F is analytic[3]: in fact, we have $\frac{d}{d\lambda}F(\lambda) = -(\lambda 1 - x)^{-2}$. So if $\mathrm{Spec}(x) = \varnothing$, then F is entire. But $\lim_{|\lambda|\to\infty} \|F(\lambda)\| = 0$, as

$$\|(\lambda - x)^{-1}\| = |\lambda^{-1}|\|(1 - x/\lambda)^{-1}\| \leq \frac{1}{|\lambda|}\frac{1}{1 - \|x/\lambda\|} \to 0.$$

By Liouville's theorem, F is constant, so $F = 0$.[4] But this is impossible. □

Theorem 1.3 (Šilov)**.** *Let M be a Banach slagebra, and let $N \subseteq M$ be a sub Banach algebra containing 1_M. If $x \in N$, then the boundary of $\mathrm{Spec}_N(x)$ is a subset of of the boundary of $\mathrm{Spec}_M(x)$.*

Remark 1.4. We always have $\mathrm{Spec}_M(x) \subseteq \mathrm{Spec}_N(x)$. This theorem gives part of the other direction.

Proof. It suffices to show that the boundary of $\mathrm{Spec}_N(x)$ is contained in $\mathrm{Spec}_M(x)$. Let $\lambda_0 \in \partial\,\mathrm{Spec}_N(x)$, and let $\{\lambda_n\} \subseteq \rho_N(x)$ with $\lambda_n \to \lambda_0$. If for some n, m, we were to have $\|(\lambda^n - x)^{-1}\| < 1/|\lambda_0 - \lambda_n|$, it would follow that $\|(\lambda_0 - x) - (\lambda_n - x)\| < 1/\|(\lambda_n - x)^{-1}\|$. Thus, $\lambda_0 - x$ is invertible in N by the lemma. This is a contradiction, so $\|(\lambda_n - x)^{-1}\| \to \infty$. Now if $\lambda_0 \notin \mathrm{Spec}_M(x)$, then $\|(\lambda - x)^{-1}\|$ is bounded for λ close enough to λ_0. This contradicts $\|(\lambda_n - x)^{-1}\| \to \infty$. □

Lemma 1.2 (Spectral radius formula)**.** $R(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

We will prove this later.

1.5 Contractivity of morphisms into C^*-algebras

Proposition 1.11. *If $N \subseteq M$ are C^*-algebras with $1_M \in N$ and $x \in N$, then $\mathrm{Spec}_N(x) = \mathrm{Spec}_M(x)$.*

Proof. Assume first that $x = x^*$. Then $\mathrm{Spec}_N(x), \mathrm{Spec}_M(x) \subseteq \mathbb{R}$. Then Šilov's theorem implies that $\mathrm{Spec}_N(x) = \mathrm{Spec}_M(x)$. For general x, invertibility of x in M implies invertibility of x^*x in M. This implies that x^*x is invertible in N, which provides a $y \in N$ such that $(yx^*)x = 1$. So x is invertible in N. □

Proposition 1.12. *Let M be a Banach involutive algebra, and let N be a C^*-algebra. Let $\pi : M \to N$ be a unital *-morphism.*[5]

[3] This is in the sense of holomorphic functional calculus.

[4] If you are uncomfortable with using Liouville's theorem when F is operator-valued, use this trick. Take any $\varphi \in M^*$. Then $\lambda \mapsto \varphi(F(\lambda))$ is analytic, entire, and $= 0$. Using Hahn-Banach, it follows that $F = 0$.

[5] This means it is an algebra homomorphism. This condition says nothing about the norm, a priori.

Proof. For $y \in N$ with $y = y^*$, we have $\|y^2\| = \|y^*y\| = \|y\|^2$. Iterating thism we get $\|y^{2^n}\|^{1/2^n} = \|y\|$. The left hand side tends to $R(y)$, so $R(y) = \|y\|$. If $x \in M$, we have $\mathrm{Spec}_N(\pi(x)) \subseteq \mathrm{Spec}_M(x)$ (since π is an algebra homomorphism, it preserves invertibility). So we get $R_N(\pi(x)) \leq R_M(x) \leq \|x\|$. So

$$\|\pi(x)\|^2 = \|\pi(x^*x)\| = R_N(\pi(x^*x)) \leq \|x^*x\| \leq \|x\|^2. \qquad \square$$